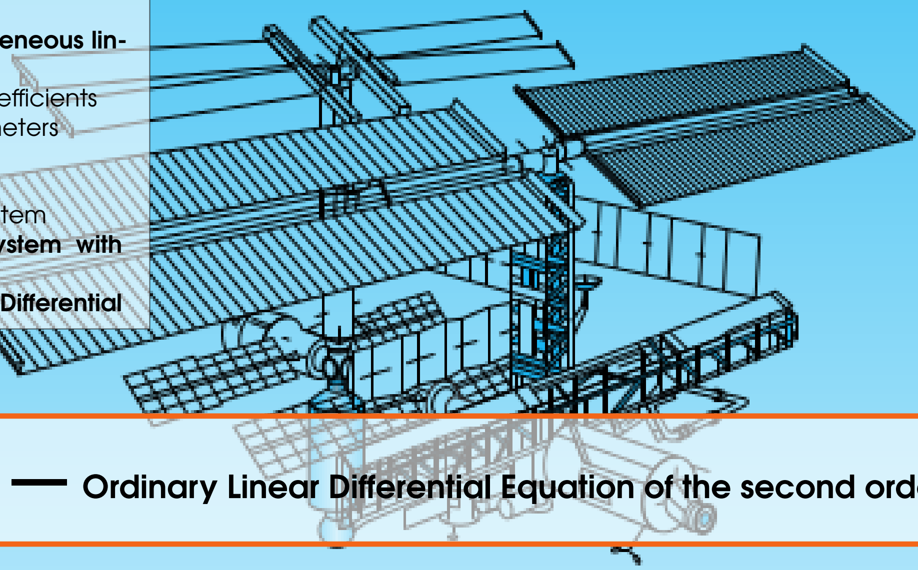

2 — Ordinary Linear Differential Equation of the second order

2.1 Homogeneous Linear Differential Equation of the second order

A second-order linear differential equation has the form

$$y'' + P(x)y' + Q(x)y = R(x) \tag{2.1}$$

where P , Q and R are continuous functions of x.
If $R(x) = 0$, for all x, then (2.1) reduces

$$y'' + P(x)y' + Q(x)y = 0 \tag{2.2}$$

and is called homogeneous. If $R(x) \neq 0$, , then (2.1) is called non-homogeneous.
Let $y_g(x, c_1, c_2)$ is the general solution of (2.2). Let y_p is a fixed particular solution of (2.1).
If y is any other solution of (2.1) then we can show that $y - y_p$ is a solution of (2.2)

$$\begin{aligned}(y-y_p)'' + P(x)(y-y_p)' + Q(x)(y-y_p) &= y_p'' - y'' + P(x)y_p' - P(x)y' + Q(x)y_p - Q(x)y \\ &= y_p'' + P(x)y_p' + Q(x)y_p - (y'' + P(x)y' + Q(x)y) = R - R = 0\end{aligned}$$

There fore, $y - y_p$ is a solution of $y'' + Py' + Qy = 0$.
Since $y_g(x, c_1, c_2)$ is the general solution of (2.2),

$$\Longrightarrow \quad y - y_p = y_g(x, c_1, c_2) \quad \Longrightarrow \quad y = y_p + y_g(x, c_1, c_2)$$

Theorem 2.1.1 — (Principle of supper position) If $y_1(x)$ and $y_2(x)$ are any solution of (2.2) then $c_1y_1(x) + c_2y_2(x)$ is also a solution of (2.2) for any constant c_1 & c_2

Proof. Since y_1 and y_2 are solution of (2.2) we have

$$y_1'' + Py_1' + Qy_1 = 0 \text{ and } y_2'' + Py_2' + Qy_2 = 0$$

Let $y = c_1y_1 + c_2y_2$. We want to show y is solution of (2.2).

$$\begin{aligned} & (c_1y_1 + c_2y_2)'' + P(c_1y_1 + c_2y_2)' + Q(c_1y_1 + c_2y_2) = c_1y_1'' + c_2y_2'' + Pc_1y_1' + Pc_2y_2' + Qc_1y_1 + Qc_2y_2 \\ \Rightarrow \quad & c_1(y_1'' + Py_1' + Qy_1) + c_2(y_2'' + Py_2' + Qy_2) \\ \Rightarrow \quad & c_1.0 + c_2.0 = 0 \end{aligned}$$

Therefore, $c_1y_1 + c_2y_2$ is also a solution of (2.2) ∎

R Super position principle in general does’t hold for non-homogeneous and non-linear.

Example 2.1
1. $y_1 = 1+\cos x$ and $y_2 = 1+\sin x$ are solutions of the non-homogeneous differential equation $y''+y = 1$ but their linear combination $y_1 + y_2 = 2+\cos x+\sin x$ is not the solution.
2. $y_1 = x^2$ and $y_2 = 1$ are the solutions of the non-linear DE $yy''-xy' = 0$ but their linear combination $y_1 + y_2 = x^2 + 1$ is not the solution.

Linear independence and Wronskian

Definition 2.1.1 If $y_1,\ y_2,\ \dots y_n$ are functions in an interval I and if each function possesses (n-1) derivatives on this interval then the **Wronskian** of the n function is

$$W(x) = W(y_1,\ y_2,\ \dots y_n) = \begin{vmatrix} y_1(x) & y_2(x) & \dots & y_n(x) \\ y_1'(x) & y_2'(x) & \dots & y_n'(x) \\ y_1''(x) & y_2''(x) & \dots & y_n''(x) \\ \vdots & \vdots & \vdots & \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \dots & y_n^{(n-1)}(x) \end{vmatrix}$$

In particular, for two differentiable functions $y_1(x)$ and $y_2(x)$ the Wronskian is defined as

$$W(x) = W(y_1,\ y_2) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x)y_2'(x) - y_1'(x)y_2(x) \tag{2.3}$$

Definition 2.1.2 A collection of function $\{y_i(x)\}_{i=1}^n$ is linearly independent on (a, b) if $\sum\limits_{i=1}^{n} c_i y_i = 0,\ \forall x \in (a,\ b)$ then $c_i = 0,\ \ (i = 0,\ 1, \dots n)$ otherwise $\{y_i(x)\}_{i=1}^n$ is called linearly dependent.

If $W(y_1,\ y_2) \neq 0$ then the function $y_1(x)$ and $y_2(x)$ are linearly independent and if $W(y_1,\ y_2) = 0$ then they are linearly dependent.

Definition 2.1.3 A set of a linearly independent solutions is called **fundamental set**

Theorem 2.1.2 Let $y_1(x)$ and $y_2(x)$ are linearly independent solution of the homogeneous equation

$$y''+P(x)y'+Q(x)y = 0 \tag{2.4}$$

on the interval [a, b] then $c_1y_1 + c_2y_2$ is the general solution of (2.4).

Corollary 2.1.3 If y_1 and y_2 are any two solution of (2.4) on (a, b) then their Wronskian $W = W(y_1,\ y_2)$ is either identically zero or never zero on [a, b]

Corollary 2.1.4 If y_1 and y_2 are any two solution of (2.4) on (a, b) then they are linearly dependent on this interval if and only if their Wronskian $W = W(y_1,\ y_2) = y_1y_2 - y_2y_1'$ is identically zero.

■ **Example 2.2** Show that $y = c_1 \sin x + c_2 \cos x$ is the general solution of $y'' + y = 0$ on any interval. Find the particular solution for which $y(0) = 2$ & $y'(0) = 3$ ■

2.2 The use of a known solution to find another (Reduction order)

Let $y'' + P(x)y' + Q(x)y = 0$ If y_1 and y_2 are linearly independent solution of (2.4), then the general solution is $y = c_1y_1 + c_2y_2$. If y_1 is a solution then cy_1 is also a solution of (2.4). Replace c by a variable v and let $y_2 = vy_1$.
Assume that y_2 is also a solution of (2.4)

$$y_2'' + Py_2' + Qy_2 = 0$$

To find v,

$$\begin{aligned}
y_2' = vy_1' + v'y_1 \quad &\text{and} \quad y_2'' = vy_1'' + 2v'y_1' + v''y_1 \\
y_2'' + Py_2' + Qy_2 \quad &= \quad vy_1'' + 2v'y_1' + v''y_1 + P(vy_1' + v'y_1) + Q(vy_1) \\
&= \quad v(y_1'' + Py_1' + Qy_1) + v'(2y_1' + py_1) + v''y_1 \\
&= \quad v''y_1 + v'(2y_1' + py_1) = 0 \\
\Longrightarrow v''y_1 + v'(2y_1' + py_1) = 0 \quad &\Longrightarrow \quad \frac{v''}{v'} = \frac{-2y_1'}{y_1} - P
\end{aligned}$$

Integrating

$$\begin{aligned}
\ln v' \quad &= \quad -2\ln y_1 - \int P(x)dx \quad \Rightarrow \quad v' = \frac{1}{y_1^2}e^{-\int P(x)dx} \\
\therefore \quad & v = \int \frac{1}{y_1^2}e^{-\int P(x)dx}dx \\
\Rightarrow \quad & y_2 = vy_1 = y_1 \int \frac{1}{y_1^2}e^{-\int P(x)dx}dx
\end{aligned}$$

■ **Example 2.3** Let $y_1 = x$ is a solution of $x^2y'' + xy' - y = 0$. Find the general solution. ■

Solution: $x^2y'' + xy' - y = 0 \;\Rightarrow\; y'' + \frac{1}{x}y'' - \frac{1}{x^2}y = 0, \;\; p(x) = \frac{1}{x}, \;\; y_2 = vy_1$

$$\begin{aligned}
v \;&=\; \int \frac{1}{y_1^2}e^{-\int P(x)dx}dx = \int \frac{1}{x^2}e^{-\int \frac{1}{x}dx}dx = \int \frac{1}{x^2}e^{-\ln x}dx = \int \frac{1}{x^3}dx = -\frac{1}{2x^2} \\
\therefore \;\; & y_2 = vy_1 = -\frac{1}{2x^2}.x = -\frac{1}{2x}
\end{aligned}$$

The general solution is $y = c_1x + c_2x^{-1}$

Exercise 2.1 Find the general solution of
(a) $y'' + y = 0, \;\; y_1 = \sin x$ (b) $y'' - y = 0, \;\; y_1 = e^x$
(c) $xy'' + 3y' = 0, \;\; y_1 = 1$ (d) $(1 - x^2)y'' - 2xy' + 2y = 0, \;\; y_1 = x$ ■

Answer:

a $y = c_1 \sin x + c_2 \cos x$
b $y = c_1e^x + c_2e^{-x}$
c $y = c_1 + c_2x^{-2}$
d $y = c_1x + c_2\left(\frac{x}{2}\ln\left(\frac{1+x}{1-x}\right) - 1\right)$

2.3 Homogeneous Differential Equation with constant coefficient

The special case of $y''+p(x)y'+q(x)y=0$ for which $p(x)$ and $q(x)$ are constants

$$y''+py'+qy=0 \tag{2.5}$$

Let $y=e^{mx}$ be possible solution of (2.5)
$y'=me^{mx},\quad y''=m^2e^{mx}$

$$\begin{aligned} & m^2e^{mx}+pme^{mx}+qe^{mx}=0 \quad \Rightarrow \quad (m^2+pm+q)e^{mx}=0 \\ \Rightarrow \quad & m^2+pm+q=0 \quad \rightarrow \text{ This equation is called auxilary/characterstics } equation \end{aligned}$$

The two roots m_1 and m_2

$$m_1=\frac{-p+\sqrt{p^2-4q}}{2},\quad m_2=\frac{-p-\sqrt{p^2-4q}}{2}$$

Case 1: Distinct two real roots $(p^2-4q>0)$.
We have two solutions e^{m_1x} and e^{m_2x}. (Let m_1 and m_2 solution for characterstics equation)

$$\tfrac{e^{m_1x}}{e^{m_2x}}=e^{(m_1-m_2)x} \text{ is not constant} \quad \Rightarrow \quad e^{m_1x} \text{ and } e^{m_2x} \text{ are linearly independent.}$$

The general solution is $y=c_1e^{m_1x}+c_2e^{m_2x}$
Case 2: If $p^2-4q=0$ (One solution)
$y=e^{mx}$ is a solution where $m=\dfrac{-p}{2}$
Let $y_1=e^{-\frac{p}{2}x}$, then $y_2=vy_1$

$$\Rightarrow \quad y_2=y_1\int\frac{1}{y_1^2}e^{-\int p\,dx}dx=e^{-\frac{p}{2}x}\int\frac{1}{y_1^2}e^{-px}dx=e^{-\frac{p}{2}x}\int\frac{1}{e^{-px}}e^{-px}dx=xe^{-\frac{p}{2}x}$$

The general solution is $y=c_1y_1+c_2y_2 \quad \Rightarrow \quad y=c_1e^{-\frac{p}{2}x}+c_2xe^{-\frac{p}{2}x}$
Case 3: If $p^2-4q<0$. In this case m_1 and m_2 can be written as $a\pm ib$

$$\begin{aligned} e^{m_1x} &= e^{(a+ib)x}=e^{ax}(\cos bx+i\sin bx),\quad e^{m_2x} = e^{(a-ib)x}=e^{ax}(\cos bx-i\sin bx) \\ \Rightarrow \quad e^{m_1x}+e^{m_2x} &= 2e^{ax}\cos bx,\quad e^{m_1x}-e^{m_2x} = 2ie^{ax}\sin bx \\ \therefore \quad y &= e^{ax}(c_1\cos bx+c_2\sin bx) \end{aligned}$$

Example 2.4 Solve the following
(a) $\dfrac{d^2y}{dx^2}-3\dfrac{dy}{dx}-4y=0$ (b) $2y''-3y'=0$ (c) $\dfrac{d^2y}{dx^2}+8\dfrac{dy}{dx}+16y=0$ (d) $\dfrac{d^2y}{dx^2}+y=0$

Exercise 2.2 1. Find the general solution of

(a) $y''-5y'-14y=0$
(b) $y''+3y'+3y=0$
(c) $y''+10y'+25y=0$
(d) $4y''-5y'=0, y(-2)=0, y'(-2)=7$
(e) $y''+14y'+49y=0,\quad y(-4)=-1,\ y'(-4)=5$

2.3.1 Cauchy-Euler equation

A linear differential equation of the form

$$ax^2\frac{d^2y}{dx^2}+bx\frac{dy}{dx}+cy=0 \tag{2.6}$$

where the coefficients a,b,c are constants, is known as a Cauchy-Euler equation.
Let $y=x^m$ be possible solution of (2.6)

$$\begin{aligned} ax^2\frac{d^2y}{dx^2}+bx\frac{dy}{dx}+cy &= ax^2m(m-1)x^{m-2}+bxmx^{m-1}+cx^m=0 \\ &\implies (am(m-1)+bm+c)x^m=0 \\ &\implies am(m-1)+bm+c=0,\ \ x^m\neq 0 \\ &\implies am^2+(b-a)m+c=0 \end{aligned} \tag{2.7}$$

CASE I : ***DISTINCT REAL ROOTS:*** Let m_1 and m_2 denote the real roots of ((2.7)) such that $m_1\neq m_2$. Then $y_1=x^{m_1}$ and $y_2=x^{m_2}$ form a fundamental set of solutions. Hence the general solution is

$$y=c_1x^{m_1}+c_2x^{m_2}$$

CASE II : ***REPEATED REAL ROOTS:*** If the roots of (2.7) are repeated (that is, $m_1=m_2$), then we obtain only one solution—namely, yx^{m_1} . When the roots of the quadratic equation $am^2+(b-a)m+c=0$ are equal, the discriminant of the coefficients is necessarily zero. It follows from the quadratic formula that the root must be $m_1=-\frac{b-a}{2a}$
Now we can construct a second solution y_2 , using reduction of order. We first write the Cauchy-Euler equation in the standard form

$$\frac{d^2y}{dx^2}+\frac{b}{ax}\frac{dy}{dx}+\frac{c}{ax^2}y=0$$

Hence, $P(x)=\frac{b}{ax} \implies \int\frac{b}{ax}=\frac{b}{a}\ln x$. Thus,

$$\begin{aligned} y_2 &= y_1\int\frac{1}{y_1^2}e^{-\int P(x)dx}dx=x^{m_1}\int\frac{1}{x^{2m_1}}e^{-\frac{b}{a}\ln x}dx \\ &= x^{m_1}\int x^{-2m_1}x^{-\frac{b}{a}}dx=x^{m_1}\int x^{\frac{b-a}{a}}x^{-\frac{b}{a}}dx \\ &= x^{m_1}\int\frac{1}{x}dx=x^{m_1}\ln x \end{aligned}$$

The general solution is then

$$y=c_1x^{m_1}+c_2x^{m_1}\ln x$$

CASE II : ***CONJUGATE COMPLEX ROOTS:*** If the roots of (2.7) are the conjugate pair

$$m_1=\alpha+i\beta, m_2=\alpha-i\beta,$$

then the general solution is

$$y=x^\alpha\left[c_1\cos(\beta\ln x)+c_2\sin(\beta\ln x)\right]$$

Example 2.5 Solve
(a) $x^2y''+3xy'+10y=0$
(b) $2x^2y''+10xy'+8y=0$
(c) $x^2y''+2xy'-12y=0$

2.4 Methods for solving non homogeneous linear differential equations

2.4.1 Method of Undetermined Coefficients

Consider

$$y'' + p(x)y' + q(x)y = R(x) \tag{2.8}$$

if $y_g(x)$ (the general solution of the associated homogenous equation) is know and y_p is a particular solution of (2.8) then

$$y = y_g(x) + y_p(x)$$

is the general solution of (2.8).
Now let us see how to found y_p with some special cases where

- the coefficients p and q are constants and
- $R(x)$ is a constant k, a polynomial function, an exponential function e^{ax} , a sine or cosine function $\sin bx$ or $\cos bx$, or finite sums and products of these functions.

The procedure for finding y_p is called the method of undetermined coefficients.

- If $R(x) = e^{ax}$ then take $y_p = Ae^{ax}$, where A is the undetermined coefficients and a is not roots of the auxiliary equation $m^2 + pm + q = 0$.
  Hence, $A = \dfrac{1}{a^2 + pa + q}, \quad a^2 + pa + q \neq 0$
  - If a is a single roots of the auxiliary equation $m^2 + pm + q = 0$, then take $y_p = Axe^{ax}$.
    Thus $A = \dfrac{1}{2a + p}, \quad 2a + p \neq 0$
  - If a is a double roots of the auxiliary equation $m^2 + pm + q = 0$, then take $y_p = Ax^2e^{ax}$.
    Thus $A = \dfrac{1}{2}$
- If $R(x) = \sin bx$ then take $y_p = A\sin bx + B\cos bx$,
  The undetermined coefficients A and B can how be computed by substituting and equating the resulting coefficients of $sinbx$ and $cosbx$.
- If $R(x) = a_0 + a_1x + a_x^2 + \ldots + a_nx^n$, take $y_p = A_0 + A_1x + A_1x^2 + \ldots + A_nx^n$

R If any y_{pi} contains terms that duplicate terms in y_g , then that y_{pi} must be multiplied by x^n , where n is the smallest positive integer that eliminates that duplication.

Example 2.6 Find the general solution of

a $y'' + 3y' - 10y = 6e^{4x}$
b $y'' + 4y = 3\sin x$
c $y'' - 2y' + 5y = 25x^2 + 12$

Exercise 2.3 Find the general solution of

(a) $y'' - 4y' + 4y = e^{2x}$
(b) $y'' + 4y = 3\cos 2x$
(c) $y'' + 4y = \sin x + \sin 2x$
(d) $y'' + y = 4x + 10\sin x, \quad y(\pi) = 0, \; y'(\pi) = 2$
(e) $y'' + 2y' + 4y = 8x^2 + 12e^{-x}$
(f) $y'' + 2y' + 4y = 8x^2 + 12e^{-x} + 10\sin 3x$

2.4.2 Method of variation of parameters

Techniques for determining a particular solution of the non homogeneous equation

$$y'' + py' + qy = R(x)$$

Let $y = c_1 y_1(x) + c_2 y_2(x)$ be the general solution of the corresponding homogeneous equations. Now we replace c_1 & c_2 by a known function v_1 & v_2

$$\begin{aligned} y(x) &= v_1 y_1 + v_2 y_2 \\ y'(x) &= v_1' y_1 + v_1 y_1' + v_2' y_2 + v_2 y_2' \\ &= (v_1' y_1 + v_2' y_2) + (v_1 y_1' + v_2 y_2') \\ \text{Let } v_1' y_1 + v_2' y_2 = 0 & \\ \Rightarrow\ y' &= v_1 y_1' + v_2 y_2' \\ y'' &= v_1' y_1' + v_1 y_1'' + v_2' y_2' + v_2 y_2'' \\ &= (v_1' y_1' + v_2' y_2') + v_1 y_1'' + v_2 y_2'' \end{aligned}$$

Substituting y, y', and y'' in the given equation we get

$$v_1(y_1'' + py_1' + qy_1) + v_2(y_2'' + py_2' + qy_2'') + v_1' y_1' + v_2' y_2' = R(x)$$

$$\Rightarrow \begin{cases} v_1' y_1' + v_2' y_2' = R(x) \\ v_1' y_1 + v_2' y_2 = 0 \end{cases} \Rightarrow \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} 0 \\ R(x) \end{pmatrix}$$

$$\Rightarrow\ v_1' = \frac{-y_2 R(x)}{W(y_1, y_2)} \ \&\ v_2' = \frac{y_1 R(x)}{W(y_1, y_2)}$$

$$\Rightarrow\ v_1 = \int \frac{-y_2 R(x)}{W(y_1, y_2)} dx, \quad v_2 = \int \frac{y_1 R(x)}{W(y_1, y_2)} dx$$

$$\therefore\ y_p = y_1 \int \frac{-y_2 R(x)}{W(y_1, y_2)} dx + y_2 \int \frac{y_1 R(x)}{W(y_1, y_2)} dx$$

■ **Example 2.7** Find the particular solution of $y'' + y = \csc x$ ■

Solution: The corresponding homogeneaous equation is

$$\begin{aligned} & y'' + y = 0 \\ \Rightarrow\ & y_g = c_1 \sin x + c_2 \cos x \\ \Rightarrow\ & y_1 = \sin x, y_2 = \cos x \Rightarrow\ W(y_1, y_2) = y_1 y_2' - y_2 y_1 = -\sin^2 x - \cos^2 x = -1 \end{aligned}$$

$$\begin{aligned} v_1 &= \int \frac{-y_2 R(x)}{W(y_1, y_2)} dx \\ &= \int \frac{-\cos x \csc x}{-1} dx = \int \frac{\cos x}{\sin x} = \ln(\sin x) \\ v_2 &= \int \frac{y_1 R(x)}{W(y_1, y_2)} dx \\ &= \int \frac{\sin x \csc x}{-1} dx = \int -dx = -x \\ \therefore\quad & y_p = v_1 y_1 + v_2 y_2 = \sin x \ln(\sin x) - x \cos x \end{aligned}$$

Example 2.8 Find the general solution of

$$y'' + 5y' + 6y = e^{-x}$$

Solution: The characteristics equation of the corresponding homogeneous DE is

$$m^2 + 5m + 6 = 0$$

Then the solution of the corresponding homogeneous equation is

$$y_g = c_1 e^{-3x} + c_2 e^{-2x}$$

Using variation of parameter with $y_1 = e^{-3x}, y_2 = e^{-2x}$ and $W = \begin{vmatrix} e^{-3x} & e^{-2x} \\ -3e^{-3x} & -2e^{-2x} \end{vmatrix} = e^{-5x}$.

Thus, we get

$$y_p = \frac{1}{2} e^{-x}$$

The general solution is:

$$y = c_1 e^{-3x} + c_2 e^{-2x} + \frac{1}{2} e^{-x}$$

Example 2.9 Solve $x^2 y'' - 3xy' + 3y = 2x^4 e^x$

Solution: Since the equation is non-homogeneous, we first solve the associated homogeneous equation. From the auxiliary equation $(m-1)(m-3) = 0$ we find $y_g = c_1 x + c_2 x^3$.

The given differential equation can be written in the form

$$y'' - \frac{3}{x} y' + \frac{3}{x^2} y = 2x^2 e^x.$$

Using variation of parameter, with $y_1 = x, y_2 = x^3$, and $W(y_1,\ y_2) = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix} = 2x^3$.

$$\begin{aligned}
v_1 &= \int \frac{-y_2 R(x)}{W(y_1, y_2)} dx = \int \frac{-x^3 (2x^2 e^x)}{2x^3} dx \\
&= -\int x^2 e^x dx = -x^2 e^x + 2x6^x - 2e^x \\
v_2 &= \int \frac{y_1 R(x)}{W(y_1, y_2)} dx = \int \frac{x(2x^2 e^x)}{2x^3} \\
&= \int e^x dx = e^x \\
\therefore y_p &= v_1 y_1 + v_2 y_2 = (-x^2 e^x + 2x6^x - 2e^x)(x) + (e^x)(x^3) = 2x^2 e^x - 2xe^x
\end{aligned}$$

The general solution is

$$y = y_g + y_p = c_1 x + c_2 x^3 + 2x^2 e^x - 2xe^x$$

Exercise 2.4 Find the general solution of

(a) $y'' - 4y' + 4y = e^{2x}$

(b) $y'' + 4y = \sec 2x; y(0) = 1, y'(0) = 2$

(c) $y'' - 2y' + y = e^x \ln x, x > 0$

(d) $x^2 y'' - xy' + y = 2x$

(e) $x^2 y'' - 2xy' + 2y = x^4 e^x$

(f) $x^2 y'' + xy' - y = \ln x$

2.5 System of Differential equation

Definition 2.5.1 A system of DE of the form

$$\begin{aligned}
\frac{dx_1}{dt} &= a_{11}(t)x_1(t)+a_{12}(t)x_2(t)+\ldots+a_{1n}(t)x_n(t)+f_1(t)\\
\frac{dx_2}{dt} &= a_{21}(t)x_1(t)+a_{22}(t)x_2(t)+\ldots+a_{2n}(t)x_n(t)+f_2(t)\\
&\vdots \qquad\qquad\qquad \vdots\\
\frac{dx_n}{dt} &= a_{n1}(t)x_1(t)+a_{n2}(t)x_2(t)+\ldots+a_{nn}(t)x_n(t)+f_n(t)
\end{aligned} \tag{2.9}$$

where the $a_{ij}(t) and f_i(t)$ are specified functions on an interval I, is called a first-order linear system. If $f_1 = f_2 = \ldots = f_n = 0$, then the system is called homogeneous. Otherwise, it is called nonhomogeneous.

Example 2.10 An example of a nonhomogeneous first-order linear system is

$$\begin{aligned}
\frac{dx_1}{dt} &= e^t x_1 + t^2 x_2 + \sin t\\
\frac{dx_1}{dt} &= t x_1 + 3x_2 - \cos t
\end{aligned}$$

The associated homogeneous system is

$$\begin{aligned}
\frac{dx_1}{dt} &= e^t x_1 + t^2 x_2\\
\frac{dx_1}{dt} &= t x_1 + 3x_2
\end{aligned}$$

■

Definition 2.5.2 By a solution to the system (2.9) on an interval I we mean an ordered n-tuple of functions $x_1(t)$, $x_2(t)$, ..., $x_n(t)$, which, when substituted into the left-hand side of the system, yield the right-hand side for all t in I.

Definition 2.5.3 Solving the system (2.9) subject to n auxiliary conditions imposed at the same value of the independent variable is called an initial-value problem (IVP). Thus, the general form of the auxiliary conditions for an IVP is:

$$x_1(t_0) = \alpha_1, x_2(t_0) = \alpha_2, \ldots, x_n(t_0) = \alpha_n,$$

where α_1, α_2, ..., α_n are constants.

2.5.1 Homogeneous Linear System

consider the homogeneous linear system

$$\begin{aligned}
\frac{dx}{dt} &= a_{11}(t)x(t)+a_{12}(t)y(t)\\
\frac{dy}{dt} &= a_{21}(t)x(t)+a_{22}(t)x(t)
\end{aligned} \tag{2.10}$$

Theorem 2.5.1 If the homogeneous linear system (2.10) has two solution

$$\begin{cases} x = x_1(t) \\ y = y_1(t) \end{cases} \quad \text{and} \quad \begin{cases} x = x_2(t) \\ y = y_2(t) \end{cases} \tag{2.11}$$

on $[a,\ b]$, then

$$\begin{cases} x = c_1x_1(t) + c_2x_2(t) \\ y = c_1y_1(t) + c_2y_2(t) \end{cases} \tag{2.12}$$

is also a solution of (2.10) on $[a,\ b]$ for arbitrary constants c_1 and c_2 and this solution (2.12) is the general solution of (2.10) if

$$\begin{vmatrix} x_1(t) & x_2(t) \\ y_1(t) & y_2(t) \end{vmatrix} \tag{2.13}$$

dose not vanish on $[a,\ b]$

Example 2.11 Show that

$$\begin{cases} x = -2e^{5t} \\ y = e^{5t} \end{cases}, \quad \text{and} \quad \begin{cases} x = 4e^{-t} \\ y = e^{-t} \end{cases}$$

is a solution to

$$\begin{aligned} x' &= x - 8y \\ y' &= -x + 3y \end{aligned}$$

on $(-\infty,\ \infty)$. Find the general solution of this system and obtain the particular solution for which

$$x(0) = 0, \ \ y(0) = 6$$

2.5.2 Non-homogeneous Linear System

Theorem 2.5.2 If the two solutions (2.11) of the homogeneous system (2.10) are linearly independent on $[a,\ b]$ and if

$$\begin{cases} x = x_p(t) \\ y = y_p(t) \end{cases}$$

is any particular solution of the non-homogeneous system

$$\begin{aligned} \frac{dx}{dt} &= a_{11}(t)x(t) + a_{12}(t)y(t) + f_1(t) \\ \frac{dy}{dt} &= a_{21}(t)x(t) + a_{22}(t)x(t) + f_2(t) \end{aligned} \tag{2.14}$$

on $[a,\ b]$ then

$$\begin{cases} x = c_1x_1(t) + c_2x_2(t) + x_p(t) \\ y = c_1y_1(t) + c_2y_2(t) + y_p(t) \end{cases}$$

is the general solution of (2.14) on $[a,\ b]$

2.6 Operator method for Linear System with constant coefficients

Consider the linear system of

$$\begin{aligned} \frac{dx}{dt} &= a_{11}x(t) + a_{12}y(t) + f_1(t) \\ \frac{dy}{dt} &= a_{21}x(t) + a_{22}x(t) + f_2(t) \end{aligned}$$

This system can be written in the equivalent form

$$(D - a_{11})x \; - \; a_{12}y \; = \; f_1(t) \tag{2.15}$$
$$-a_{21}x \; + \; (D - a_{22})y \; = \; f_2(t) \tag{2.16}$$

where D is the differential operator $\frac{d}{dt}$. The idea behind the solution technique is that we can now easily eliminate y between these two equations by operating on equation (2.15) with $D - a_{22}$, multiplying equation (2.16) by a_{12}, and adding the resulting equations. This yields a second-order constant coefficient linear differential equation for x only. Substituting the expression thereby obtained for x into equation (2.15) will then yield y.

Example 2.12 Solve the IVP

$$\begin{aligned} x' &= x + 2y \\ y' &= 2x - 2y, \quad x(0) = 1,\ y(0) = 0 \end{aligned}$$

Solution: Rewriting the system in operator form as

$$(D-1)x \; - \; 2y \; = \; 0 \tag{2.17}$$
$$-2x \; + \; (D+2)y \; = \; 0 \tag{2.18}$$

To eliminate y between these two equations, we first operate on equation (2.17) with $D+2$ to obtain

$$(D+2)(D-1)x - 2(D+2)y = 0$$

Adding twice equation (2.18) to this equation eliminates y and yields

$$(D+2)(D-1)x - 4x = 0 \implies (D^2 + D - 6)x = 0$$

This constant coefficient DE has auxiliary polynomial

$$m^2 + m - 6 = 0 \implies (m+3)(m-2) = 0 \implies m = -3 \text{ or } m = 2$$

Hence,

$$x = c_1e^{-3t} + c_2e^{2t}$$

We now determine y. From equation (2.17), we have

$$\begin{aligned} y &= \frac{1}{2}(D-1)x = \frac{1}{2}(Dx - x) \\ &= \frac{1}{2}\left(\frac{d}{dt}(c_1e^{-3t} + c_2e^{2t}) - (c_1e^{-3t} + c_2e^{2t})\right) \\ &= \frac{1}{2}\left(-4c_1e^{-3t} + c_2e^{2t}\right) \end{aligned}$$

Hence, the solution to the given system of DE is

$$\begin{cases} x = c_1e^{-3t} + c_2e^{2t} \\ y = \frac{1}{2}\left(-4c_1e^{-3t} + c_2e^{2t}\right) \end{cases}$$

where c_1 and c_2 are arbitrary constants. Imposing the two initial conditions yields the following equations for determining c_1 and c_2:

$$c_1 + c_2 = 1, \quad -4c_1 + c_2 = 0 \quad \Rightarrow \quad c_1 = \frac{1}{5}, \quad c_2 = \frac{4}{5}$$

Hence, the particular solution is

$$\begin{cases} x = \frac{1}{5}\left(e^{-3t} + 4e^{2t}\right) \\ y = \frac{2}{5}\left(e^{2t} - e^{-3t}\right) \end{cases}$$

Exercise 2.5 Solve

a.
$$\begin{aligned} \frac{dx}{dt} + 4x + 3y &= t \\ \frac{dy}{dt} + 2x + 5y &= e^t \end{aligned}$$

b.
$$\begin{aligned} \frac{dx}{dt} &= x + 2y + t - 1 \\ \frac{dy}{dt} &= 3x + 2y - 5t - 2 \end{aligned}$$

c.
$$\begin{aligned} Dx - 3y &= 6a\sin t \\ 3x + Dy &= 0 \end{aligned}$$

subject to $x(0) = a, \quad y(0) = 0$

2.7 Applications of Second-Order Differential Equations

2.7.1 Spring/Mass System

HOOKE'S LAW: states that the spring itself exerts a restoring force F opposite to the direction of elongation and proportional to the amount of elongation s. i.e.,

$$F = ks$$

where k is a constant of proportionality called the **spring constant**.

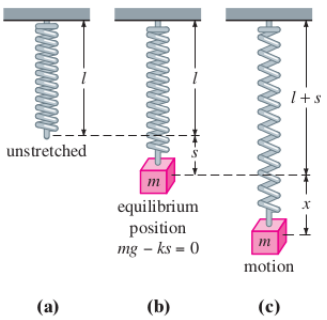

Figure 2.1: Spring/mass system

NEWTON'S SECOND LAW After a mass m is attached to a spring, it stretches the spring by an amount s and attains a position of equilibrium at which its weight $W = mg$ is balanced by the restoring force ks. If the mass is displaced by an amount x from its equilibrium position, the restoring force of the spring is then $k(x+s)$.
Assuming that there are no retarding forces acting on the system and assuming that the mass vibrates free of other external forces '**free motion**' we can equate Newton's second law with the net, or resultant, force of the restoring force and the weight:

$$m\frac{d^2x}{dt^2} = -k(x+s)+mg = -kx+mg-ks = -kx \tag{2.19}$$

The negative sign in (2.19) indicates that the restoring force of the spring acts opposite to the direction of motion. Furthermore, we adopt the convention that displacements measured below the equilibrium position are positive. See Figure 2.2

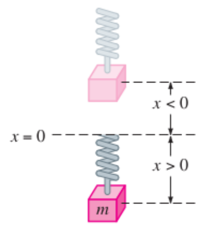

Figure 2.2: Direction below the equilibrium position is positive.

Differential equation of free undamped motion : By dividing (2.19) by the mass m, we obtain the second-order differential equation

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \tag{2.20}$$

where $\omega^2 = \frac{k}{m}$. Equation (2.20) is said to describe simple harmonic motion or free undamped motion. If the system starts at $t = 0$ with an initial position x_0 and initial velocity x_1, we have initial condition's $x(0) = x_0, \quad$ and $\; x'(0) = x_1$.
Thus, the general solution of (2.20) is

$$x(t) = c_1 \cos\omega t + c_2 \sin\omega t \tag{2.21}$$

The natural frequency is $f = \frac{\omega}{2\pi}$ and the **period** of motion is $T = \frac{1}{f} = \frac{2\pi}{\omega}$. The number $\omega = \sqrt{\frac{k}{m}}$ (measured in radians per second) is called the circular frequency of the system.
Equation (2.21) can be re-expressed as

$$x(t) = A\cos(\omega t - \phi)$$

where $A = \sqrt{c_1^2 + c_2^2}$ is **Amplitude** and
$\phi = \tan^{-1}(c_2/c_1)$, is **phase angle**. Where $\sin\phi = \frac{c_2}{A}$, $\cos\phi = \frac{c_2}{A}$
Differential equation of free damped motion: In the study of mechanics, damping forces acting on a body are considered to be proportional to a power of the instantaneous velocity. In particular, we shall assume throughout the subsequent discussion that this force is given by a

constant multiple of $\frac{dx}{dt}$. When no other external forces are impressed on the system, it follows from Newton's second law that

$$m\frac{d^2x}{dt^2} = -kx - \beta\frac{dx}{dt} \tag{2.22}$$

where β is a positive damping constant and the negative sign is a consequence of the fact that the damping force acts in a direction opposite to the motion.
Dividing (2.22) by the mass m, we find that the differential equation of free damped motion is

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0 \tag{2.23}$$

where $2\lambda = \frac{\beta}{m}$, $\omega^2 = \frac{k}{m}$.
Case I: If $\lambda^2 - \omega^2 > 0$. The system is said to be **overdamped** because the damping coefficient β is large when compared to the spring constant k.
The corresponding solution of (2.23) is

$$x(t) = e^{-\lambda t}\left(c_1 e^{\sqrt{\lambda^2-\omega^2}t} + c_2 e^{-\sqrt{\lambda^2-\omega^2}t}\right)$$

This equation represents a smooth and nonoscillatory motion.
Case II: If $\lambda^2 - \omega^2 = 0$. The system is said to be **critically damped** because any slight decrease in the damping force would result in oscillatory motion. The general solution of (2.23) is

$$x(t) = e^{-\lambda t}(c_1 + c_2 t) \tag{2.24}$$

The motion is quite similar to that of an overdamped system. It is also apparent from (2.24) that the mass can pass through the equilibrium position at most one time.
Case III: If $\lambda^2 - \omega^2 < 0$. The system is said to be **underdamped**, since the damping coefficient is small in comparison to the spring constant. Thus the general solution of equation (2.23) is

$$x(t) = e^{-\lambda t}(c_1 \cos\sqrt{\lambda^2 - \omega^2}t + c_2 \sin\sqrt{\lambda^2 - \omega^2}t)$$

The motion is oscillatory; but because of the coefficient $e^{-\lambda t}$ the amplitudes of vibration $\to 0$ as $t \to \infty$

■ **Example 2.13** A spring with a mass of 2 kg has natural length 0.5 m. A force of 25.6 N is required to maintain it streched to a length of 0.7 m. If the spring is streched to a length of 0.7 m and then released with initial velocity 0, find the position of the mass at any time t. ■

■ **Example 2.14** Suppose that the spring of Example 2.13 is immersed in a fluid with damping constant $\beta = 40$. Find the position of the mass at any time t if it starts from the equilibrium position and is given a push to start it with an initial velocity of 0.6 m/s. ■

2.7.2 Electric Circuit

Consider the RLC Circuit below
Kirchhoff's Law The algebric sum of the voltage drops in a simple closed circuit is zero. The voltage drop across the resistor, capacitor and inductor are given RI, $\frac{1}{c}q$, and $L\frac{dI}{dt}$ respactively. Hence

$$RI + L\frac{dI}{dt} + \frac{1}{c}q = E(t) \tag{2.25}$$

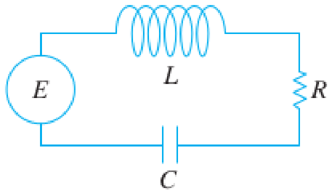

Figure 2.3: LRC series circuit.

Since $I = \dfrac{dq}{dt} \Rightarrow \dfrac{dI}{dt} = \dfrac{d^2q}{dt^2}$

$$\Rightarrow \frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{1}{cL}q = \frac{E(t)}{L} \tag{2.26}$$

The initial conditions may be $q(0) = q_0,\ \dfrac{dq}{dt}|_{t=0} = I(0) = I_0$
To obtain a differential equation for current differentiating equ (2.25) with repect to time t,

$$R\frac{dI}{dt} + L\frac{d^2I}{dt^2} + \frac{1}{c}\frac{dq}{dt} = \frac{dE(t)}{dt}$$

Since $\dfrac{dq}{dt} = I$

$$\Rightarrow \frac{d^2I}{dt^2} + \frac{R}{L}\frac{dI}{dt} + \frac{1}{cL}I = \frac{E(t)}{dt} \tag{2.27}$$

The initial conditions may be $I(0) = I_0,$ and $\dfrac{dI}{dt}|_{t=0} = \dfrac{1}{L}.$
If $E(t) = 0$, the electrical vibrations of the circuit are said to be free.
We say that the circuit is

$$\textbf{overdamped} \text{ if } R^2 - \frac{4L}{C} > 0$$
$$\textbf{critically damped} \text{ if } R^2 - \frac{4L}{C} = 0$$
$$\textbf{underdamped} \text{ if } R^2 - \frac{4L}{C} < 0$$

■ **Example 2.15** Find the charge q(t) on the capacitor in an LRC series circuit when $L = 0.25\ H$, $R = 10ohms$, $C = 0.001$ farad , $E(t) = 0,\ q(0) = q_0$ coulombs, and $I(0) = 0$. ■